AMENABLE GROUPS AND ACTIONS

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In this short handout, we introduce the concept of amenability for actions and, as a particular case, for groups. We present the first properties of such actions, as well as examples of amenable and non-amenable groups.

1. Means on a set

To introduce and study amenability, we first need to define and understand *means* on a set X.

Definition 1.1. A mean on a set X is a map $\mu\colon \mathcal{P}(X) \longrightarrow [0,1]$ so that $\mu(X) = 1$ and $\mu(A \sqcup B) = \mu(A) + \mu(B)$ for all disjoint $A, B \in \mathcal{P}(X)$.

In other words, a mean on a set X is a *finitely* additive probability measure.

Remark 1.2. As for probability measures, there are several properties inherited from the definition. We let the reader check that if μ is as in the Definition 1.1, it satisfies:

(i) $\mu(\emptyset) = 0$.
(ii) $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$, for all $A, B \in \mathcal{P}(X)$.
(iii) $\mu\left(\bigsqcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i)$ for all $A_1, \ldots, A_n \in \mathcal{P}(X)$.
(iv) $\mu(A) \leq \mu(B)$ for all $A, B \in \mathcal{P}(X)$ such that $A \subset B$.

Given a set X, we will denote by $\mathcal{M}(X)$ the set of all means on X, and we see it as a topological subspace of $\mathbb{R}^{\mathcal{P}(X)}$. Moreover we endow $\mathbb{R}^{\mathcal{P}(X)}$ with the product topology, also called the topology of point-wise convergence. Therefore a sequence $(\mu_n)_{n\in\mathbb{N}}$ of means converges to μ in $\mathbb{R}^{\mathcal{P}(X)}$ if and only if $\mu_n(A) \longrightarrow \mu(A)$ in $[0,1] \subset \mathbb{R}$, for all $A \in \mathcal{P}(X)$.

Example 1.3. Suppose $X \neq \emptyset$ and choose $x \in X$. The *Dirac mass at x* is defined as

$$\delta_x(A) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}.$$

for all $A \in \mathcal{P}(X)$. One easily checks that $\delta_x \in \mathcal{M}(X)$.

For a while, Dirac masses will be our only examples of means on a set. To prove there exists other means, more difficult to understand, we will establish some linear and topological properties of $\mathcal{M}(X)$. Note for instance that $\mathcal{M}(X)$ is *not* a vector subspace of $\mathbb{R}^{\mathcal{P}(X)}$, as it does not contain the zero function. It is also not closed under arbitrary linear combinations because of the condition $\mu(X) = 1$.

Lemma 1.4. *The set $\mathcal{M}(X)$ is convex.*

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Proof. This is a straightforward check. Let $\mu, \eta \in \mathcal{M}(X)$ and $\lambda \in [0,1]$. Since $\mu(X) = \eta(X) = 1$, we have $(\lambda\mu + (1-\lambda)\eta)(X) = \lambda + (1-\lambda) = 1$. Likewise, if $A, B \subset X$ are disjoint, then

$$\begin{aligned}(\lambda\mu + (1-\lambda)\eta)(A \sqcup B) &= \lambda\mu(A \sqcup B) + (1-\lambda)\eta(A \sqcup B) \\ &= (\lambda\mu + (1-\lambda)\eta)(A) + (\lambda\mu + (1-\lambda)\eta)(B)\end{aligned}$$

using finite additivity of μ and η. This proves the claim. □

From the lemma, the next corollary is immediate.

Corollary 1.5. *Any convex combination of means is a mean.*

Proof. A convex combination of means is of the form $\sum_{i=1}^{n} \lambda_i \mu_i$ with $\sum_{i=1}^{n} \lambda_i = 1$ and $\mu_i \in \mathcal{M}(X)$ for all $1 \le i \le n$. We conclude using induction on n and Lemma 1.4. □

Let us now turn to topological properties. By definition, a basis for the product topology on $\mathbb{R}^{\mathcal{P}(X)}$ is given by all sets of the form

$$O_1 \times \cdots \times O_n \times \prod_{\mathcal{P}(X)\setminus\{A_1,\ldots,A_n\}} \mathbb{R}$$

where $O_1, \ldots, O_n$ are open sets of $\mathbb{R}$ and $A_1, \ldots, A_n \in \mathcal{P}(X)$. A subbasis is therefore given by

$$\left\{ O \times \prod_{\mathcal{P}(X)\setminus\{A\}} \mathbb{R} \ : \ O \subset \mathbb{R} \quad \text{open}, \ A \in \mathcal{P}(X) \right\}.$$

The complement of an open set of this form is $(\mathbb{R} \setminus O) \times \prod_{\mathcal{P}(X)\setminus\{A\}} \mathbb{R}$. Hence we can construct closed subsets of $\mathbb{R}^{\mathcal{P}(X)}$ by fixing finitely many $A_1, \ldots, A_n \in \mathcal{P}(X)$, choosing any closed subset $C \subset \mathbb{R}^{\{A_1,\ldots,A_n\}} \simeq \mathbb{R}^n$, and considering

$$C \times \prod_{\mathcal{P}(X)\setminus\{A_1,\ldots,A_n\}} \mathbb{R}.$$

This observation makes a lot easier the proof of the following fact.

Proposition 1.6. *The space $\mathcal{M}(X)$ is compact for the product topology on $\mathbb{R}^{\mathcal{P}(X)}$.*

Proof. Firstly, note that in fact $\mathcal{M}(X) \subset [0,1]^{\mathcal{P}(X)}$ and the latter is compact by Tychonoff's theorem. It is thus enough to prove that $\mathcal{M}(X)$ is closed in $[0,1]^{\mathcal{P}(X)}$. By expanding the definition, we have

$$\begin{aligned}\mathcal{M}(X) &= \{\mu \in [0,1]^{\mathcal{P}(X)} : \mu(X) = 1, \ \mu(A \sqcup B) = \mu(A) + \mu(B) \ \forall A, B \in \mathcal{P}(X)\} \\ &= \{\mu \in [0,1]^{\mathcal{P}(X)} \ : \ \mu(X) = 1\} \\ &\cap \bigcap_{A,B\in\mathcal{P}(X),\ A\cap B=\emptyset} \left\{ \mu \in [0,1]^{\mathcal{P}(X)} \ : \ \mu(A \sqcup B) = \mu(A) + \mu(B) \right\}.\end{aligned}$$

All sets in this writing of $\mathcal{M}(X)$ are closed by the observation above: the first one corresponds to the choice $A_1 = X$, $C = \{1\}$. The second one, for A and B fixed with

$A \cap B = \emptyset$, corresponds to $A_1 = A$, $A_2 = B$, $A_3 = A \sqcup B$ and $C = \{(x, y, z) \in \mathbb{R}^3 \mid x+y = z\}$. Therefore $\mathcal{M}(X)$ is closed in $[0, 1]^{\mathcal{P}(X)}$ as an intersection of closed sets. This concludes our proof. □

Recall that if $(x_n)_{n\in\mathbb{N}}$ is a sequence in a topological space X, it has an accumulation point $y \in X$ if for every open set $U \subset X$ with $y \in U$ and for every $N \in \mathbb{N}$, there exists $n \geq N$ so that $x_n \in U$. Recall furthermore that if X is a compact topological space, then any sequence has an accumulation point. Indeed, suppose that $(x_n)_{n\in\mathbb{N}}$ is a sequence with no accumulation point in X. Then for any $y \in X$ there is an open subset U_y with $y \in U_y$ and containing only finitely many terms of the sequence $(x_n)_{n\in\mathbb{N}}$. The collection $\{U_y\}_{y\in Y}$ is then an open covering of X, and by compactness there exists $y_1, \ldots, y_m \in X$ with $X \subset U_{y_1} \cup \cdots \cup U_{y_m}$. This implies that our sequence $(x_n)_{n\in\mathbb{N}}$ contains only finitely many terms, a contradiction.

Though we must be careful. In a compact topological space a sequence does not have necessarily a convergent subsequence. Topological spaces with this property are called *sequentially* compact, and are usually not compact. However, sequential compactness is equivalent to compactness for metric spaces, and more generally for metrisable spaces.

2. First examples of amenable groups

Now is the time for defining amenable groups and actions.

Definition 2.1. Let G be a group and X a set. An action $G \curvearrowright X$ is called amenable if there exists $\mu \in \mathcal{M}(X)$ so that $\mu(gA) = \mu(A)$ for $A \in \mathcal{P}(X)$ and all $g \in G$.

When such a μ exists, we call it a *left invariant* mean.

Definition 2.2. A group G is amenable if the left multiplication action of G on itself is amenable.

That is, a group G is amenable if there exists $\mu \in \mathcal{M}(G)$ so that $\mu(gA) = \mu(A)$ for all $A \subset G$ and $g \in G$. As an exercise, show that if such a left invariant mean exists, then G also has a *right* invariant mean.

Here is our central example of an amenable action.

Example 2.3. Let G be a group, and $X \neq \emptyset$ a finite G−space. Let $\mu := \frac{1}{|X|}\sum_{x\in X} \delta_x$, and fix $g \in G$. As $\delta_x(gA) = \delta_{g^{-1}x}(A)$ for all $A \subset X$ and $x \in X$, μ is a left invariant mean:

$$\mu(gA) = \frac{1}{|X|}\sum_{x\in X} \delta_x(gA) = \frac{1}{|X|}\sum_{x\in X} \delta_{g^{-1}x}(A) = \frac{1}{|X|}\sum_{x\in X} \delta_{g^{-1}x}(A) = \mu(A).$$

This proves that $G \curvearrowright X$ is amenable. Note that no assumption on G is required.

Corollary 2.4. *Finite groups are amenable.*

Proof. If G is a finite group, it suffices to apply Example 2.3 with $X = G$ itself. □

Let $f : X \longrightarrow Y$ be any map. It induces a map $f_* : \mathcal{M}(X) \longrightarrow \mathcal{M}(Y)$ defined as $f_*(\mu)(A) := \mu(f^{-1}(A))$ for all $\mu \in \mathcal{M}(X)$ and $A \in \mathcal{P}(Y)$. Hence an action $G \curvearrowright X$

automatically induces an action $G \curvearrowright \mathcal{M}(X)$, and one can restate Definition 2.1 as follows: $G \curvearrowright X$ is amenable if $\mathcal{M}(X)^G \neq \emptyset$, where

$$\mathcal{M}(X)^G := \{\mu \in \mathcal{M}(X) : \forall g \in G,\ g\mu = \mu\}$$

stands for the set of fixed points of the action $G \curvearrowright \mathcal{M}(X)$.

Remark 2.5. Let $f\colon X \longrightarrow Y$. For the product topologies on $\mathcal{M}(X) \subset \mathbb{R}^{\mathcal{P}(X)}$ and $\mathcal{M}(Y) \subset \mathbb{R}^{\mathcal{P}(Y)}$, f_* is continuous. Indeed, if $\mathcal{O} \times \prod_{\mathcal{P}(Y)\setminus\{A\}} \mathbb{R}$ is an element of the subbasis for the topology on $\mathcal{M}(Y)$, with $\mathcal{O} \subset \mathbb{R}$ open and $A \in \mathcal{P}(Y)$, we have

$$\begin{aligned} f_*^{-1}\left(\mathcal{O} \times \prod_{\mathcal{P}(Y)\setminus\{A\}} \mathbb{R}\right) &= \left\{\mu \in \mathcal{M}(X) \ :\ f_*(\mu) \in \mathcal{O} \times \prod_{\mathcal{P}(Y)\setminus\{A\}} \mathbb{R}\right\} \\ &= \{\mu \in \mathcal{M}(X) \ :\ f_*(\mu)(A) \in \mathcal{O}\} \\ &= \{\mu \in \mathcal{M}(X) \ :\ \mu(f^{-1}(A)) \in \mathcal{O}\} \\ &= \mathcal{O} \times \prod_{\mathcal{P}(X)\setminus\{f^{-1}(A)\}} \mathbb{R} \end{aligned}$$

which is open in $\mathcal{M}(X)$. This shows that f_* is continuous as claimed. Moreover, f_* preserves convex combinations, in the sense that $f_*(\lambda\mu+(1-\lambda)\eta) = \lambda f_*(\mu)+(1-\lambda)f_*(\eta)$ for all $\mu, \eta \in \mathcal{M}(X)$ and all $\lambda \in [0,1]$.

Explicitly, for a finite group G, $\mu(A) = \dfrac{|A|}{|G|}$ is a left invariant mean. The next result gives an example of an amenable group without any explicit formula for an invariant mean.

Theorem 2.6. *The group $\mathbb{Z}$ is amenable.*

Proof. For $n \geq 1$, consider $\mu_n := \dfrac{1}{n}\sum_{j=1}^{n} \delta_j$. By Corollary 1.5, $\mu_n \in \mathcal{M}(\mathbb{Z})$ for every $n \geq 1$. By Proposition 1.6, $\mathcal{M}(\mathbb{Z})$ is compact, so $(\mu_n)_{n\geq 1}$ has an accumulation point $\mu \in \mathcal{M}(\mathbb{Z})$. We now prove μ is a left invariant mean for the action $\mathbb{Z} \curvearrowright \mathbb{Z}$, *i.e.* we must show that $g\mu = \mu$ for all $g \in \mathbb{Z}$. Writing $\mathbb{Z} = \langle u \rangle$ multiplicatively, it is enough to prove that $u\mu = \mu$. First, note that if $A \subset \mathbb{Z}$, then

$$u\delta_j(A) = \delta_j(u^{-1}A) = \begin{cases} 1 & \text{if } j \in u^{-1}A \\ 0 & \text{if } j \notin u^{-1}A \end{cases} = \begin{cases} 1 & \text{if } uj = j+1 \in A \\ 0 & \text{if } uj = j+1 \notin A \end{cases} = \delta_{j+1}(A)$$

so that $u\delta_j = \delta_{j+1}$. It follows that $u\mu_n = \dfrac{1}{n}\sum_{j=2}^{n+1} \delta_j$, and thus $u\mu_n - \mu_n = \frac{1}{n}(\delta_{n+1} - \delta_1)$ for all $n \geq 1$. This implies that

$$u\mu_n - \mu_n \longrightarrow 0 \tag{2.1}$$

as $n \to \infty$, in $\mathbb{R}^{\mathcal{P}(\mathbb{Z})}$. If $u\mu \neq \mu$, we can separate them by disjoint neighbourhoods U and V (because $\mathbb{R}^{\mathcal{P}(\mathbb{Z})}$ is Hausdorff, as a product of Hausdorff spaces). Since μ is an accumulation point of $(\mu_n)_{n\in\mathbb{N}}$ and $u\mu$ is an accumulation point of $(u\mu_n)_{n\in\mathbb{N}}$, we can find infinitely many terms of the sequence $(\mu_n)_{n\in\mathbb{N}}$ in U and infinitely many terms of $(u\mu_n)_{n\in\mathbb{N}}$ in V. Since they are disjoint, this contradicts (2.1). Thus $u\mu = \mu$, and this finishes the proof. □

Remark 2.7. The fact that $u\mu$ is an accumulation point of $(u\mu_n)_n$ follows from the continuity of

$$\begin{aligned} \varphi_u \colon \mathcal{M}(\mathbb{Z}) &\longrightarrow \mathcal{M}(\mathbb{Z}) \\ \mu &\longmapsto u\mu \end{aligned}$$

and the continuity of φ_u follows from Remark 2.5.

As recalled above, for general topological spaces compactness is not necessarily equivalent to sequential compactness, and indeed the sequence $(\mu_n)_{n\geq 1}$ involved in the above proof does not have a convergent subsequence, although $\mathcal{M}(\mathbb{Z})$ is compact. Suppose for a contradiction that $(\mu_{n_k})_{k\in\mathbb{N}}$ is such a subsequence, and denote $\mu \in \mathcal{M}(\mathbb{Z})$ its limit. For $r \geq 1$, let $A_r := \{-r, \dots, r\}$. Then

$$\mu_{n_k}(A_r) = \begin{cases} 1 & \text{if } n_k \leq r \\ \frac{r}{n_k} & \text{if } n_k > r \end{cases}$$

which tends to 0 as $k \to \infty$. It implies that $\mu(A_r) = \lim\limits_{k\to\infty} \mu_{n_k}(A_r) = 0$ for all $r \geq 1$. This forces $\mu(\mathbb{Z}) = 0$, which is absurd. In particular, $\mathcal{M}(\mathbb{Z})$ is not sequentially compact, and therefore not metrisable either, nor second countable.

3. Free actions and amenability

We prove here our first stability result about the class of amenable groups: it is closed when taking subgroups. The next lemma is the first step in that direction. Recall that a set map $f \colon X \longrightarrow Y$ between two $G-$sets is called *a $G-$map* if it commutes to the action of G, *i.e.* if $f(gx) = gf(x)$ for all $g \in G$ and $x \in X$. Note that compositions and inverses of $G-$maps are $G-$maps as well.

Lemma 3.1. *Let X, Y be two $G-$sets, and $f \colon X \longrightarrow Y$ be a $G-$map. If $G \curvearrowright X$ is amenable, then $G \curvearrowright Y$ is amenable.*

Proof. Recall that $f \colon X \longrightarrow Y$ induces $f_* \colon \mathcal{M}(X) \longrightarrow \mathcal{M}(Y)$. Since $G \curvearrowright X$ is amenable, there exists $\mu \in \mathcal{M}(X)^G$. We then prove that $f_*(\mu) \in \mathcal{M}(Y)$ is a fixed point for the action $G \curvearrowright \mathcal{M}(Y)$. Since f is a $G-$map, one has $gf_*(\mu) = f_*(g\mu)$ for all $g \in G$ (check!), and it follows that

$$gf_*(\mu) = f_*(g\mu) = f_*(\mu)$$

for all $g \in G$ since μ is $G-$invariant. This finishes the proof. □

This result implies that an amenable group always acts amenably.

Corollary 3.2. *If G is amenable, then any action $G \curvearrowright X$ with $X \neq \emptyset$ is amenable.*

Proof. Since $X \neq \emptyset$, choose $x_0 \in X$. The map $f \colon G \longrightarrow X$, $f(g) := gx_0$ is a $G-$map, and the action $G \curvearrowright G$ is amenable by hypothesis. Lemma 3.1 implies then that $G \curvearrowright X$ is amenable, as claimed. □

We now wish to determine whether the converse is true, *i.e.* if a group acts amenably on a non-empty set, is the group amenable? In fact, it is not hard to see that with no further assumption, this statement is false. It suffices to consider for instance an action of a non-amenable group (an example of such a group will be provided in the next section) on a finite set. Such an action is amenable by the Example 2.3.

Definition 3.3. A group action $G \curvearrowright X$ is free if $\mathrm{Stab}_G(x) = \{e\}$ for all $x \in X$.

With words, a group action is free if it has no fixed points, *i.e.* $X^G = \emptyset$. It turns out it is a sufficient condition to ensure amenability of the acting group.

Proposition 3.4. *If $G \curvearrowright X$ is an amenable free action, then G is amenable.*

Proof. Let $G \curvearrowright X$ be an amenable free action, and let $R \subset X$ be a set of representatives of the G−orbits: we choose exactly one representative per orbit.
We define

$$\begin{aligned} f\colon G \times R &\longrightarrow X \\ (g,r) &\longmapsto gr \end{aligned}$$

Since R is a set of representatives, and since orbits form a partition of X, f is surjective. In fact, it is also injective. If (g,r), $(g',r') \in G \times R$ are such that $gr = g'r'$ then $(g')^{-1}gr = r'$, so r and r' are in the same orbit, and so $r = r'$ since we chose one element in each orbit. But this implies $(g')^{-1}gr = r$, which means $(g')^{-1}g \in \mathrm{Stab}_G(r) = \{e\}$. Thus $g = g'$, and f is injective. Now we consider $G \times R$ as a G−set with the left multiplication action on G and the trivial action on R. For this action on $G \times R$, f is a G−map, since

$$\forall \gamma \in G,\ \forall (g,r) \in G \times R,\ f(\gamma(g,r)) = f(\gamma g, r) = (\gamma g)r = \gamma(gr) = \gamma f(g,r).$$

We have almost all ingredients to apply Lemma 3.1, but f goes in the wrong direction. We then consider its inverse $f^{-1}\colon X \longrightarrow G \times R$, which is a G−map. Likewise, the projection $p_G\colon G \times R \longrightarrow G$ is a G−map. Therefore the composition $p_G \circ f^{-1}\colon X \longrightarrow G$ is a G−map. Since $G \curvearrowright X$ is amenable, Lemma 3.1 applies, and $G \curvearrowright G$ is an amenable action, which means exactly that G is amenable. □

This result is the key statement to conclude on amenability of subgroups of amenable groups.

Corollary 3.5. *Let G be an amenable group. Then any subgroup of G is amenable. In particular if G contains a non-amenable subgroup, then G is not amenable.*

Proof. Suppose G is amenable, and let $H \leqslant G$. Choose $\mu \in \mathcal{M}(G)^G$. In particular, $\mu \in \mathcal{M}(G)^H$ so that the action $H \curvearrowright G$ by left multiplication is amenable. It is straightforward to check that this action is free. Thus H is amenable by Proposition 3.4. □

4. Free groups are not amenable

In this section we construct the simpliest example of an infinite family of non-amenable groups.

Let S be any set. The *free group* on S, denoted F_S, is the unique group satisfying the following universal property: for any group G, for any map $f\colon S \longrightarrow G$, there exists a unique group homomorphism $\tilde{f}\colon F_S \longrightarrow G$ extending f. In other words, there exists a unique group homomorphism $\tilde{f}$ making the following diagram commute:

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\iota"'] & G \\ F_S \arrow[ur, "\tilde{f}"] & \end{tikzcd}$$

Here $\iota\colon S \hookrightarrow F_S$ is the natural inclusion of S in F_S.

Remark 4.1. The free group F_S depends only on $|S|$, up to isomorphism. We will write F_d if $|S| = d$.

Note that F_0 is the trivial group, and F_1 is infinite cyclic, so $F_1 \cong \mathbb{Z}$. For $n \geq 2$, F_n is not abelian. Free groups play a central role in group theory, and more details on their construction and properties can be found in [1, 2]. One particularly important result about them is the so called *Nielsen-Schreier theorem*, stating that any subgroup of a free group is free.

For us free groups provides the other part of the spectrum, opposite to finite groups and $\mathbb{Z}$, as they are not amenable.

Theorem 4.2. *The group F_2 is not amenable.*

Proof. Suppose for a contradiction that there exists a left invariant mean μ on F_2. Write $F_2 = \{e\} \sqcup A_+ \sqcup A_- \sqcup B_+ \sqcup B_-$, where A_+ (resp. A_-) consists of reduced words starting with an a (resp. a^{-1}) and B_+ (resp. B_-) consists of reduced words starting with a b (resp. b^{-1}). Since the second letter of an element of A_+ can be an a, a b or a b^{-1}, multiplying this element by a^{-1} produces an element either of A_+, B_+ or B_-. Then

$$a^{-1}A_+ = \{e\} \sqcup A_+ \sqcup B_+ \sqcup B_-.$$

Properties of μ then imply

$$\mu(A_+) = \mu(a^{-1}A_+) = \mu(\{e\} \sqcup A_+ \sqcup B_+ \sqcup B_-) = \mu(\{e\}) + \mu(A_+) + \mu(B_+) + \mu(B_-)$$

and erasing $\mu(A_+)$ of both sides leaves us with $\mu(\{e\}) + \mu(B_+) + \mu(B_-) = 0$. Since μ takes positive values, this forces $\mu(\{e\}) = \mu(B_+) = \mu(B_-) = 0$. Likewise, we get $\mu(A_+) = \mu(A_-) = 0$. We conclude that

$$\begin{aligned} 1 = \mu(F_2) &= \mu(\{e\} \sqcup A_+ \sqcup A_- \sqcup B_+ \sqcup B_-) \\ &= \mu(\{e\}) + \mu(A_+) + \mu(A_-) + \mu(B_+) + \mu(B_-) \\ &= 0 \end{aligned}$$

which is absurd. Therefore such a μ cannot exist. □

Corollary 4.3. *For any $d \geq 2$, F_d is not amenable.*

Proof. As F_d contains F_2 for any $d \geq 2$, and the latter is not amenable, Corollary 3.5 gives the conclusion. □

Remark 4.4. The converse of Corollary 3.5 is not true: consider for instance the non-amenable group $G = F_2 = \langle a, b | \emptyset \rangle$ with the amenable subgroup $H = \langle a \rangle \cong \mathbb{Z}$.

It is then a first sufficient criterion to establish the non-amenability of some discrete groups: prove that the group in question contains a subgroup isomorphic to F_2. The main standard result to achieve this is the following, usually known as the *Ping-Pong Lemma*.

Lemma 4.5. *Let G be a group acting on a non-empty set X. Let $\Gamma_1, \Gamma_2 \leqslant G$ be subgroups of G such that $|\Gamma_1| \geq 3$, $|\Gamma_2| \geq 2$. Suppose there exists disjoint non-empty subsets X_1, $X_2 \subset X$ such that*

$$\forall \gamma \in \Gamma_1 \setminus \{e\},\ \gamma \cdot X_2 \subset X_1 \quad \text{and} \quad \forall \gamma \in \Gamma_2 \setminus \{e\},\ \gamma \cdot X_1 \subset X_2.$$

Then the subgroup $\Gamma := \langle \Gamma_1, \Gamma_2 \rangle$ is free in G.

Proof. We just need to check every reduced word $w = \gamma_1 \dots \gamma_n \in \Gamma$ is not the trivial word. For this we distinguish several cases.

(i) The length n of w is odd and $\gamma_1, \gamma_n \in \Gamma_1$. Since $X_2 \neq \emptyset$, we pick $x \in X_2$. By hypothesis, $\gamma_n \cdot x \in X_1$, so $\gamma_{n-1} \cdot \gamma_n \cdot x \in X_2$ (note that γ_{n-1} indeed lie in Γ_2, otherwise our initial word w was not reduced). Continuing this process, we finally arrive at $\gamma_2 \dots \gamma_n x \in X_2$ since $n-1$ is even. Hence, by applying our hypothesis once more, we get $w \cdot x = \gamma_1(\gamma_2 \dots \gamma_n x) \in X_1$. As X_1 and X_2 are disjoint we then have $wx \neq x$, so w cannot be the trivial word.

(ii) The length n of w is odd and $\gamma_1 \in \Gamma_2$, $\gamma_n \in \Gamma_2$. Here we can choose $\gamma \in \Gamma_1$ and consider the word $\gamma w \gamma^{-1}$. It has odd length and first and last letter in Γ_1. By (i), we then know $\gamma w \gamma^{-1} \neq e$, which implies $w \neq e$.

(iii) The length n of w is even and $\gamma_1 \in \Gamma_1$, $\gamma_n \in \Gamma_2$. We pick $\gamma \in \Gamma_1$ and we consider $\gamma w \gamma^{-1}$. By reduction its first letter is then $\gamma\gamma_1 \in \Gamma_1$, while its last letter is just $\gamma^{-1} \in \Gamma_1$. Thus $\gamma w \gamma^{-1}$ has odd length, and first and last letter in Γ_1. We may apply (i) to get $\gamma w \gamma^{-1} \neq e$, so $w \neq e$.

(iv) The length n of w is even and $\gamma_1 \in \Gamma_2$, $\gamma_n \in \Gamma_1$. As before we choose an arbitrary $\gamma \in \Gamma_1$ and we look at $\gamma w \gamma^{-1}$. It has odd length after reduction, and its first and last letter lie in Γ_1. Hence $w \neq e$. This concludes the fourth case, and also our proof. □

Example 4.6. Consider the action of $\mathrm{SL}_2(\mathbb{Z})$ on the plane $\mathbb{R}^2$, and the two matrices

$$A := \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad B := \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Observe that by setting

$$X_1 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \ : \ |x| > |y| \right\},\ X_2 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \ : \ |x| < |y| \right\}$$

we obtain $AX_2 \subset X_1$ and $BX_1 \subset X_2$. Indeed if we take $x, y \in \mathbb{R}$ with $|x| < |y|$ and we let A act on the vector formed by x and y the result is a vector whose components are $x + 2y$ and y. Using the second triangle inequality we see that

$$|x + 2y| \geq ||2y| - |x|| \geq |2y| - |x| > 2|y| - |y| = |y|$$

meaning the vector we got falls in X_1. The arguing is the same for $BX_1 \subset X_2$. Of course X_1 and X_2 are non-empty and disjoint, so we may apply Lemma 4.5 with $\Gamma_1 := \langle A \rangle$ and $\Gamma_2 := \langle B \rangle$. Both are infinite because A and B have infinite order. Hence $\langle A, B \rangle$ is a free group in $\mathrm{SL}_2(\mathbb{Z})$, namely a non-abelian free group on two generators F_2. It follows that $\mathrm{SL}_2(\mathbb{Z})$ is not amenable.

References

[1] K. Conrad, “Why words problems are hard”, Expository paper. Available online at https://kconrad.math.uconn.edu/blurbs/.

[2] Tullio Cecherrini-Silberstein, Michele D’Adderio, Topics in Groups and Geometry, Springer monographs in Mathematics, 2021.